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Selecting Good Normal Regression Models : an Empirical Bayes Approach[1]

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SUMMARY

This paper deals with the problem of selecting all good normal regression models using the parametric empirical Bayes approach. The average of k linear loss functions is used as the loss function for the selection problem, where k is the number of regression models under consideration for the selection problem. Mimicking the behaviour of a Bayes selection rule, an empirical Bayes selection rule is constructed. Also, the corresponding asymptotic optimality is investigated. It is shown that under certain conditions on the independent variables of the regression models, the regret risk of the proposed empirical Bayes selection rule converges to 0 with a rate of order k^{-1}.

Key words : Asymptotic optimality, Bayes selection rule, Empirical Bayes, Good population, Normal regression model, Rate of convergence.

1. Introduction

Consider k independent normal populations $\pi_1 = N(\theta_1, \sigma^2)$, ..., $\pi_k = N(\theta_k, \sigma^2)$ with unknown means $\theta_1, \ldots, \theta_k$ and a common unknown variance σ^2. For a given control value θ_0, population π_i is said to be good if $\theta_i \geq \theta_0$, and bad otherwise. The problem of selecting all good normal populations has been extensively studied in the literature. To mention some earlier papers, Paulson [8] and Gupta and Sobel [5] have studied problems of selecting a subset containing all good populations using some natural selection rules. Randles and Hollander [10], Miescke [7] and Gupta and Miescke [2] have derived optimal selection rules via the Γ-minimax and minimax approaches. Huang [6] has derived Bayes selection rules to partition normal populations with respect to a control. The reader is referred to Gupta and Panchapakesan ([3], [4]) for an overview on this research area. The paper, aims at deriving selection rules for selecting all good normal populations via the parametric empirical Bayes approach.

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Let $\Omega = \{ \underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k) \mid \theta_i \in R, i = 1, \ldots, k \}$ be the parameter space. Let $\underset{\sim}{a} = (a_1, \ldots, a_k)$ be an action, where $a_i = 0, 1$; $i = 1, \ldots, k$. When action $\underset{\sim}{a}$ is taken, it means that population π_i is selected as good if $a_i = 1$ and excluded as bad if $a_i = 0$. Consider the following loss function :

$$L(\underset{\sim}{\theta}, \underset{\sim}{a}) = \frac{1}{k} \sum_{i=1}^{k} L_i(\theta_i, a_i) \tag{1.1}$$

where, for each $i = 1, \ldots, k$,

$$L_i(\theta_i, a_i) = a_i(\theta_0 - \theta_i) I_{(-\infty, \theta_0)}(\theta_i) + (1 - a_i)(\theta_i - \theta_0) I_{(\theta_0, \infty)}(\theta_i) \tag{1.2}$$

where I_S denotes the indicator function of the set S. In (1.2), the first term is the loss of selecting π_i as good while $\theta_i < \theta_0$, and the second term is the loss of not selecting π_i when π_i is good.

For each $i = 1, \ldots, k$, let $Y_{i1}, \ldots, Y_{im}$ be a sample of size $m\ (m \geq 2)$ from population $\pi_i = N(\theta_i, \sigma^2)$. It is assumed that θ_i is a realization of a random variable Θ_i, which has a $N(\underset{\sim}{x}'_i \underset{\sim}{\beta}, \tau^2)$ prior distribution, where $\underset{\sim}{x}'_i = (x_{i1}, \ldots, x_{ip})$ is a known vector, $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_p)$ is an unknown parameter vector and the variance τ^2 is unknown. The random variables $\Theta_1, \ldots, \Theta_k$ are assumed to be mutually independent. Let $\underset{\sim}{Y}_i = (Y_{i1}, \ldots, Y_{im})$, $\underset{\sim}{Y} = (\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_k)$ and let $\mathcal{Y}$ denote the sample space of $\underset{\sim}{Y}$. A selection rule $\underset{\sim}{d} = (d_1, \ldots, d_k)$ is a mapping defined on the sample space $\mathcal{Y}$ such that for each $\underset{\sim}{y} \in \mathcal{Y}$, $d_i(\underset{\sim}{y})$ is the probability of selecting π_i as a good population.

Under the preceding statistical model, the Bayes risk of the selection rule $\underset{\sim}{d}$ is

$$R(\underset{\sim}{d}) = \frac{1}{k} \sum_{i=1}^{k} R_i(d_i) \tag{1.3}$$

where

$$R_i(d_i) = \int_{\underset{\sim}{y} \in \mathcal{Y}} d_i(\underset{\sim}{y}) [\theta_0 - \varphi_i(\underset{\sim}{y}_i)] \prod_{j=1}^{k} f_j(\underset{\sim}{y}_j) d\underset{\sim}{y} + C_i \tag{1.4}$$

and

$$C_i = E[(\Theta_i - \theta_0) I_{(\theta_0, \infty)}(\Theta_i)]$$

$f_j(\underset{\sim}{y}_j)$ is the marginally joint probability density of $\underset{\sim}{Y}_j = (Y_{j1}, ..., Y_{jm})$,

$\varphi_i(\underset{\sim}{y}_i) = E[\Theta_i \mid \underset{\sim}{Y}_i = \underset{\sim}{y}_i] = (1-\alpha)\bar{y}_i + a\,\underset{\sim}{x}'_i \underset{\sim}{\beta} = \psi_i(\bar{y}_i)$ is the posterior mean of Θ_i given $\underset{\sim}{Y}_i = \underset{\sim}{y}_i$, where $\bar{y}_i = \frac{1}{m}\sum_{j=1}^{m} y_{ij}$ and $\alpha = \frac{\sigma^2}{m} \Big/ \left(\frac{\sigma^2}{m} + \tau^2\right)$.

Hence, a Bayes selection rule $\underset{\sim}{d}_G = (d_{G1}, \ldots, d_{Gk})$, which minimizes the Bayes risks among all selection rules, is given as follows :

For each $\underset{\sim}{y} \in \mathcal{Y}$ and each $i = 1, \ldots, k$,

$$d_{Gi}(\underset{\sim}{y}) = \begin{cases} 1 & \text{if } \psi_i(\bar{y}_i) \geq \theta_0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \bar{y}_i \geq [\theta_0 - \alpha\,\underset{\sim}{x}'_i \underset{\sim}{\beta}]/(1-\alpha) \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

From (1.5), one can see that for each component i, the Bayes selection rule d_{Gi} is independent of $y_j, j \neq i$, and depends on y_i only through the sample mean value $\bar{y}_i$, and is non-decreasing in $\bar{y}_i$. Hence it can also be written as $d_{Gi}(\bar{y}_i)$. The minimum Bayes risk is :

$$R(\underset{\sim}{d}_G) = \frac{1}{k}\sum_{i=1}^{k} R_i(d_{Gi}) \tag{1.6}$$

where

$$R_i(d_{Gi}) = \int_{-\infty}^{\infty} d_{Gi}(\bar{y}_i)[\theta_0 - \psi_i(\bar{y}_i)]\, g_i(\bar{y}_i)\, d\bar{y}_i + C_i \tag{1.7}$$

and $g_i(\bar{y}_i)$ is the marginal probability density of the sample mean $\bar{Y}_i = \frac{1}{m}\sum_{j=1}^{m} Y_{ij}$. It is known that marginally, $\bar{Y}_i$ follows the normal distribution $N\left(\underset{\sim}{x}'_i \underset{\sim}{\beta}, \frac{\sigma^2}{m} + \tau^2\right)$.

2. *Empirical Bayes Selection Rule*

It should be noted that the Bayes selection rule $\underset{\sim}{d}_G$ strongly depends on $\psi_i(\bar{y}_i)$, i = 1, ..., k, which are also dependent on parameters β and α. Since these parameters are unknown, the Bayes selection rule $\underset{\sim}{\tilde{d}}_G$ cannot be implemented for the selection problem at hand. In the following, the empirical Bayes approach is applied. First construct estimators for the unknown parameters $\underset{\sim}{\beta}$ and α. Then, by mimicking the behaviour of the Bayes selection rule $\underset{\sim}{d}_G$, an empirical Bayes selection rule, say $\underset{\sim}{d}^*$, is derived. The performance of the empirical Bayes selection rule $\underset{\sim}{d}^*$ will be evaluated in the next section.

For each $i = 1, \ldots, k$, let $\underset{\sim}{x}(i) = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_{i-1}, \underset{\sim}{x}_{i+1}, \ldots, \underset{\sim}{x}_k)$. It is assumed that $k > p$ and for each $i = 1, \ldots, k$, $\underset{\sim}{x}(i)$ has rank p. Let $P(i) = \underset{\sim}{x}'(i)(\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}\underset{\sim}{x}(i)$. Note that marginally $\bar{Y}_j \sim N(\underset{\sim}{x}'_j \underset{\sim}{\beta}, \frac{\sigma^2}{m} + \tau^2)$, $j = 1, \ldots, k$, and $\bar{Y}_1, \ldots, \bar{Y}_k$ are mutually independent. Let $\underset{\sim}{\bar{Y}}'(i) = (\bar{Y}_1, \ldots, \bar{Y}_{i-1}, \bar{Y}_{i+1}, \ldots, \bar{Y}_k)$. Under the normal regression model, for each $i = 1, \ldots, k$, the maximal likelihood estimator of $\underset{\sim}{\beta}$ based on $\underset{\sim}{\bar{Y}}(i)$ is :

$$\underset{\sim}{\hat{\beta}}(i) = (\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}\underset{\sim}{x}(i)\underset{\sim}{\bar{Y}}(i) \tag{2.1}$$

Next, construct estimator for $\alpha = \frac{\sigma^2}{m} \Big/ \left(\frac{\sigma^2}{m} + \tau^2\right)$. For each $j = 1, \ldots, k$, let $W_j = \sum_{l=1}^{m} (Y_{jl} - \bar{Y}_j)^2$ and $W = \sum_{j=1}^{k} W_j$. Since for each $j = 1, \ldots, k$, $\frac{W_j}{\sigma^2} \sim \chi^2(m-1)$ and $W_1, \ldots, W_k$ are iid, therefore, $\frac{W}{\sigma^2} \sim \chi^2(k(m-1))$ and $\frac{W}{mk(m-1)}$ is an unbiased estimator of $\frac{\sigma^2}{m}$. Let $V_i = \underset{\sim}{\bar{Y}}'(i)(I_{k-1} - P(i))\underset{\sim}{\bar{Y}}(i)$. It is known that $V_i \Big/ \left(\frac{\sigma^2}{m} + \tau^2\right) \sim \chi^2(k-1-p)$ and therefore, $V_i/(k-1-p)$ is an unbiased estimator of $\frac{\sigma^2}{m} + \tau^2$. Hence, it is natural to use the ratio $\frac{W}{mk(m-1)} \Big/ \left(\frac{V_i}{k-1-p}\right)$ as an estimator of α. However, when $\alpha \le 1$, it is possible that the value of the ratio is greater than one. Hence, estimate α by

$\hat{\alpha}(i) = \min\left(\frac{W}{mk(m-1)} / \left(\frac{V_i}{k-1-p}\right), 1\right)$. Then estimate the posterior mean $\psi_i(\bar{y}_i) = (1-\alpha)\bar{y}_i + \alpha \underset{\sim}{x}'_i \underset{\sim}{\beta}$ by

$$\hat{\psi}_i(\bar{y}_i) = [1 - \hat{\alpha}(i)]\,\bar{y}_i + \hat{\alpha}(i)\,\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) \tag{2.2}$$

Now, by mimicking the behaviour of the Bayes selection rule $\underset{\sim}{d}_G$, we propose an empirical Bayes selection rule $\underset{\sim}{d}^* = (d_1^*, \ldots, d_k^*)$ as follows : For each $i = 1, \ldots, k$ and $\underset{\sim}{y} \in \mathcal{Y}$,

$$d_i^*(\underset{\sim}{y}) = \begin{cases} 1 & \text{if } \psi_i(\bar{y}_i) \geq \theta_0 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Note that the empirical Bayes selection rule d_i^* depends on $\underset{\sim}{y}$ only through $\bar{y}_i$, $\hat{\alpha}(i)$ and $\hat{\underset{\sim}{\beta}}(i)$, where the latter two are functions of W, V_i and $\underset{\sim}{\bar{Y}}_{(i)}$. For fixed $\hat{\alpha}(i)$ and $\hat{\underset{\sim}{\beta}}(i)$, d_i^* is non decreasing in $\bar{y}_i$. Let P_i denote the probability measure generated by W, V_i and $\underset{\sim}{\bar{Y}}_{(i)}$, and let E_i denote the expectation taken with respect to the probability measure P_i. Note that W, V_i, $\underset{\sim}{\bar{Y}}(i)$ and $\bar{Y}_i$ are mutually independent. Based on the preceding reasoning, the empirical Bayes selection rule $\underset{\sim}{d}^*$ can be presented as :

For each $i = 1, \ldots, k$,

$$d_i^*(\bar{y}_i \mid \hat{\alpha}(i), \hat{\underset{\sim}{\beta}}(i)) = \begin{cases} 1 & \text{if } \hat{\psi}_i(\bar{y}_i) \geq \theta_0 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

The Bayes risk of the empirical Bayes selection rule $\underset{\sim}{d}^*$ can be written as:

$$R(\underset{\sim}{d}^*) = \frac{1}{k}\sum_{i=1}^{k} R_i(d_i^*) \tag{2.5}$$

where

$$\begin{aligned} R_i(d_i^*) &= E_i\left[\int_{\bar{y}_i=-\infty}^{\infty} d_i^*(\bar{y}_i \mid \hat{\alpha}(i), \hat{\underset{\sim}{\beta}}(i))\,[\theta_0 - \psi_i(\bar{y}_i)]\,g_i(\bar{y}_i)\,d\bar{y}_i\right] + C_i \\ &= \int_{\bar{y}_i=-\infty}^{\infty} P_i\{d_i^*(\bar{y}_i \mid \hat{\alpha}(i), \hat{\underset{\sim}{\beta}}(i)) = 1\}\,[\theta_0 - \psi_i(\bar{y}_i)]\,g_i(\bar{y}_i)\,d\bar{y}_i + C_i \end{aligned} \tag{2.6}$$

3. *Asymptotic Optimality*

Let $\underset{\sim}{d}$ be any selection rule and $R(\underset{\sim}{d})$ the corresponding Bayes risk. Since $\underset{\sim}{d}_G$ is the Bayes selection rule $D_i(d_i) = R_i(d_i) - R_i(d_{Gi}) \geq 0$ for each $i = 1, \ldots, k$. Hence, $D(\underset{\sim}{d}) = R(\underset{\sim}{d}) - R(\underset{\sim}{d}_G) = \frac{1}{k}\sum_{i=1}^{k} D_i(d_i) \geq 0$. $D(\underset{\sim}{d})$ is called the regret risk of the selection rule $\underset{\sim}{d}$. The regret risk $D(\underset{\sim}{d})$ is always used as a measure of performance of the selection rule $\underset{\sim}{d}$.

Definition 3.1. A selection rule $\underset{\sim}{d}$ is said to be asymptotically optimal of order $\{\varepsilon_k\}$ if $D(d) = 0(\varepsilon_k)$ where $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\lim_{k \to \infty} \varepsilon_k = 0$.

In the following the asymptotic optimality of the empirical Bayes selection rule $\underset{\sim}{d}^*$ is studied. For this purpose, it is assumed that Condition C holds.

Condition C (1) $\sum_{j=1}^{p} x_{ij}^2 < M$ for all i where M is a positive value independent of k;

(2) $\frac{1}{k}\underset{\sim}{x}\underset{\sim}{x}'$ converges to a positive definite matrix A as k tends to infinity.

Now, the regret risk of the empirical Bayes selection rule $\underset{\sim}{d}^*$ is

$$D(\underset{\sim}{d}^*) = \frac{1}{k}\sum_{i=1}^{k} D_i(d_i^*) \tag{3.1}$$

where

$$D_i(d_i^*) = E_i \int_{\bar{y}_i = -\infty}^{\infty} [\, d_i^*(\bar{y}_i \mid \hat{\alpha}(i), \hat{\underset{\sim}{\beta}}(i)) - d_{Gi}(\bar{y}_i)\,]\,[\,\theta_0 - \psi_i(\bar{y}_i)\,]\, g_i(\bar{y}_i)\, d\bar{y}_i \tag{3.2}$$

In the following, without loss of generality, it is assumed that $\theta_0 \leq \underset{\sim}{x}'_i \underset{\sim}{\beta}$.

By the definitions of d_{Gi} and d_i^*, we obtain that

$$\int_{\bar{y}_i = -\infty}^{\infty} [d_i^*(\bar{y}_i \mid \hat{\alpha}(i), \hat{\underset{\sim}{\beta}}(i)) - d_{Gi}(\bar{y}_i)]\,[\,\theta_0 - \psi_i(\bar{y}_i)\,]\, g_i(\bar{y}_i)\, d\bar{y}_i$$

$$= \int_{\bar{y}_i=-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\, I[\hat{\alpha}(i) < 1 \text{ and } \hat{\psi}_i(\bar{y}_i) \geq \theta_0]\, g_i(\bar{y}_i)\, d\bar{y}_i$$

$$+ \int_{\bar{y}_i=-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\, I[\hat{\alpha}(i) = 1 \text{ and } \psi_i(\bar{y}_i) \geq \theta_0]\, g_i(\bar{y}_i)\, d\bar{y}_i$$

$$+ \int_{\bar{y}_i=a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_o]\, I[\hat{\alpha}(i) < 1 \text{ and } \psi_i(\bar{y}_i) < \theta_0]\, g_i(\bar{y}_i)\, d\bar{y}_i$$

$$+ \int_{\bar{y}_i=a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_0]\, I[\hat{\alpha}(i) = 1 \text{ and } \hat{\psi}_i(\bar{y}_i) < \theta_0]\, g_i(\bar{y}_i)\, d\bar{y}_i$$

$$\equiv I_i + II_i + III_i + IV_i \text{ (say)} \tag{3.3}$$

where $a_i = (\theta_0 - \alpha\, \underset{\sim}{x}'_i\, \underset{\sim}{\beta})/(1-\alpha)$. Note that $a_i \leq \underset{\sim}{x}'_i\, \underset{\sim}{\beta}$. Note that $\theta_0 - \psi_i(\bar{y}_i) \geq 0$ as $\bar{y}_i < a_i$ and $\psi_i(\bar{y}_i) - \theta_0 \geq 0$ as $\bar{y}_i > a_i$. Hence,

$$E_i[II_i] \leq \int_{\bar{y}_i=-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\; g_i(\bar{y}_i)\, d\bar{y}_i\, E_i[I(\hat{\alpha}(i) = 1)]$$

$$\leq M_1 \exp\left\{-\frac{k(m-1)}{2}\left[\frac{1-\alpha}{2\alpha} - \ln\left(1 + \frac{1-\alpha}{2\alpha}\right)\right]\right\}$$

$$+ M_1 \exp\left\{-\frac{k-1-p}{2}\left[-\frac{1-\alpha}{2} - \ln\left(1 - \frac{1-\alpha}{2}\right)\right]\right\}$$

$$\leq O(k^{-1}) \tag{3.4}$$

where, the second inequality is obtained from Lemmas A2(c) and A3(a).

Similarly

$$E_i[IV_i] \leq \int_{a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i)\, d\bar{y}_i\, E_i[I(\hat{\alpha}(i) = 1)]$$

$$\leq O(k^{-1}) \tag{3.5}$$

Next, consider

$$E_i[I_i] = \int_{\bar{y}_i=-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\, g_i(\bar{y}_i)\, P_i\{\hat{\alpha}(i) < 1 \text{ and } \hat{\psi}_i(\bar{y}_i) > \theta_0\}\, d\bar{y}_i$$

For $\bar{y}_i < a_i$, by the definition of $\hat{\psi}_i(\bar{y}_i)$, we have

$P_i\{\hat{\alpha}(i) < 1 \text{ and } \hat{\psi}_i(\bar{y}_i) > \theta_0\}$

$= P_i \{ \hat{\alpha}(i) < 1$ and

$$(\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) + \hat{\alpha}(i)(\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta}) > \theta_0 - \psi_i(\bar{y}_i) \}$$

$$\leq P_i \left\{ \hat{\alpha}(i) < 1 \text{ and } (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) > \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \right\}$$

$$+ P_i \{ \hat{\alpha}(i) < 1 \text{ and } \hat{\alpha}(i)(\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta}) > \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \}$$

$$\equiv I_{i1} + I_{i2} \tag{3.6}$$

By Lemma A4(a),

$$I_{i2} \leq P_i \left\{ \underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta} > \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \right\}$$

$$\leq \frac{\sqrt{2b_i v^2}}{\sqrt{\pi}\,(\theta_0 - \psi_i(\bar{y}_i))} \exp \left\{ - \frac{(\theta_0 - \psi_i(\bar{y}_i))^2}{8 b_i v^2} \right\} \tag{3.7}$$

where $b_i = \underset{\sim}{x}'_i (\underset{\sim}{x}(i) \underset{\sim}{x}'(i))^{-1} \underset{\sim}{x}_i$ and $v^2 = \frac{\sigma^2}{m} + \tau^2$.

Also,

$$E_i[III_i] = \int_{\bar{y}_i = a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i)\, P_i \{ \hat{\alpha}(i) < 1 \text{ and } \psi_i(\bar{y}_i) < \theta_0 \} d\bar{y}_i$$

For $\bar{y}_i > a_i$, by the definition of $\hat{\psi}_i(\bar{y}_i)$, we have

$$P_i \{ \hat{\alpha}(i) < 1 \text{ and } \hat{\psi}_i(\bar{y}_i) < \theta_0 \}$$

$$\leq P_i \{ \hat{\alpha}(i) < 1 \text{ and } (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \}$$

$$+ P_i \{ \hat{\alpha}(i) < 1 \text{ and } \hat{\alpha}(i)(\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}_i' \underset{\sim}{\beta}) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \}$$

$$\equiv III_{i1} + III_{i2} \tag{3.8}$$

By Lemma A.4(b),

$$III_{i2} \leq P_i \left\{ \underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta} < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2} \right\}$$

$$\leq \frac{\sqrt{2b_i v^2}}{\sqrt{\pi}\,(\psi_i(\bar{y}_i) - \theta_0)} \exp \left\{ - \frac{(\psi_i(\bar{y}_i) - \theta_0)^2}{8 b_i v^2} \right\} \tag{3.9}$$

Combining the preceding results yields that

$$E\,[I_i + I\,I\,I_i] \le A_1 + A_2 + A_3 \tag{3.10}$$

where

$$A_1 = \int_{\bar{y}_i=-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\, g_i(\bar{y}_i)\, P_i\, \{\hat{\alpha}(i) < 1, (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) > \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\}\, d\bar{y}_i$$

$$A_2 = \int_{\bar{y}_i = a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i)\, P_i\, \{\hat{\alpha}(i) < 1, (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\}\, d\bar{y}_i$$

and

$$A_3 = \int_{\bar{y}_i=-\infty}^{\infty} \frac{\sqrt{2b_i v^2}}{\sqrt{\pi}} \exp\left\{- \frac{(\theta_0 - \psi_i(\bar{y}_i))^2}{8 b_i v^2}\right\} g_i(\bar{y}_i)\, d\bar{y}_i$$

By noting that $\bar{Y}_i \sim N(\underset{\sim}{x}'_i \underset{\sim}{\beta}, v^2)$ and by Lemma A.5,

$$A_3 = \sqrt{\frac{8v^2}{\pi}}\, \frac{b_i}{\sqrt{(1-\alpha)^2 + 4b_i}} \exp\left\{\frac{(\underset{\sim}{x}'_i \underset{\sim}{\beta}) - \theta_0)^2}{2v^2\,[(1-\alpha)^2 - 4b_i]}\right\}$$

$$= O(k^{-1}) \tag{3.11}$$

Therefore, it suffices to consider the asymptotic behaviour of A_1 and A_2.

For $\bar{y}_i < a_i$, $c(\bar{y}_i) \equiv \dfrac{\theta_0 - \psi_i(\bar{y}_i)}{2(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)} = \dfrac{\theta_0 - \underset{\sim}{x}'_i \underset{\sim}{\beta}}{2(x' \underset{\sim}{\beta} - \bar{y}_i)} + \dfrac{1-\alpha}{2}$ is decreasing in $\bar{y}_i$, since $\theta_0 \le \underset{\sim}{x}'_i \underset{\sim}{\beta}$. Thus, for $\bar{y}_i < a_i$, $c(\bar{y}_i) > c(a_i) = 0$, and by Lemma A.2,

$$P_i\left\{\hat{\alpha}(i) < 1 \text{ and } (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) > \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\right\}$$

$$\le P_i\{\hat{\alpha}(i) - \alpha > c(\bar{y}_i)\}$$

$$\le \exp\left\{- \frac{k(m-1)}{2}\, h_1(c(\bar{y}_i), \alpha)\right\} + \exp\left\{- \frac{k-1-p}{2}\, h_2(c(\bar{y}_i), \alpha)\right\} \tag{3.12}$$

where

$$h_1(c, \alpha) = \frac{c}{2\alpha} - \ln\left(1 + \frac{c}{2\alpha}\right) \text{ and } h_2(c, \alpha) = -\frac{c}{2(\alpha + c)} - \ln\left(\frac{c}{2(\alpha + c)}\right)$$

By substituting (3.12) into A_1 and by Lemma A.7, we obtain

$$A_1 \le \int_{\bar{y}_i = -\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)] \, g_i(\bar{y}_i) \exp\left\{-\frac{k(m-1)}{2} h_1(c(\bar{y}_i), \alpha)\right\} d\bar{y}_i$$

$$+ \int_{\bar{y}_i = -\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)] \, g_i(\bar{y}_i) \exp\left\{-\frac{k-1-p}{2} h_2(c(\bar{y}_i), \alpha)\right\} d\bar{y}_i$$

$$= O(k^{-1}) \tag{3.13}$$

Finally, we need to take care of A_2. Note that $a_i \le \underset{\sim}{x}'_i \underset{\sim}{\beta}$ since it is assumed that $\theta_0 \le \underset{\sim}{x}'_i \underset{\sim}{\beta}$. Thus,

$$A_2 = \int_{a_i}^{\underset{\sim}{x}'_i \underset{\sim}{\beta}} [\psi_i(\bar{y}_i) - \theta_0] \, g_i(\bar{y}_i) P_i\left\{\hat{\alpha}(i) < 1, (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\right\} d\bar{y}_i$$

$$+ \int_{\underset{\sim}{x}'_i \underset{\sim}{\beta}}^{\infty} [\psi_i(\bar{y}_i) - \theta_0] \, g_i(\bar{y}_i) P_i\left\{\hat{\alpha}(i) < 1, (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\right\} d\bar{y}_i$$

$$\equiv A_{21} + A_{22} \tag{3.14}$$

For $a_i < \bar{y}_i < \underset{\sim}{x}'_i \underset{\sim}{\beta}$, $\theta_0 - \psi_i(\bar{y}_i) < 0$, and

$$P_i\left\{\hat{\alpha}(i) < 1, (\hat{\alpha}(i) - \alpha)(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i) < \frac{\theta_0 - \psi_i(\bar{y}_i)}{2}\right\}$$

$$\le P_i\{\hat{\alpha}(i) - \alpha < c(\bar{y}_i)\}$$

$$= 0 \text{ if } \alpha + c(\bar{y}_i) \le 0$$

So, in the following, consider only those $\bar{y}_i \in (a_i, \underset{\sim}{x}'_i \underset{\sim}{\beta})$ such that $\alpha + c(\bar{y}_i) > 0$, which is equivalent to that $\bar{y}_i < \dfrac{\alpha \underset{\sim}{x}'_i \underset{\sim}{\beta} + \theta_0}{1 + \alpha} \equiv e_i$. Note that $a_i < e_i < \underset{\sim}{x}'_i \underset{\sim}{\beta}$. For $\bar{y}_i \in (a_i, e_i)$, by Lemma A.2, we have

$$P_i\left\{\hat{\alpha}(i)<1, (\hat{\alpha}(i)-\alpha)(\underset{\sim}{x}'_i\underset{\sim}{\beta}-\bar{y}_i) < \frac{\theta_0-\psi_i(\bar{y}_i)}{2}\right\}$$

$$\leq P_i\{\hat{\alpha}(i)-\alpha<c(\bar{y}_i)\} \tag{3.15}$$

$$\leq \exp\left\{-\frac{k(m-1)}{2}h_1(c(\bar{y}_i),\alpha)\right\}+\exp\left\{-\frac{k-1-p}{2}h_2(c(\bar{y}_i),\alpha)\right\}$$

Replacing the inequality of (3.15) into A_{21}, and by Lemma A.8(a), (b), we obtain:

$$A_{21}=O(k^{-1}) \tag{3.16}$$

For $\bar{y}_i > \underset{\sim}{x}'_i\underset{\sim}{\beta}$,

$$P_i\left\{\hat{\alpha}(i)<1, (\hat{\alpha}(i)-\alpha)(\underset{\sim}{x}'_i\underset{\sim}{\beta}-\bar{y}_i) < \frac{\theta_0-\psi_i(\bar{y}_i)}{2}\right\}$$

$$\leq P_i\{\hat{\alpha}(i)-\alpha>c(\bar{y}_i)\}$$

$$= 0 \text{ if } \alpha+c(\bar{y}_i)>1$$

So, consider only those $\bar{y}_i > \underset{\sim}{x}'_i\underset{\sim}{\beta}$ such that $0<c(\bar{y}_i)<1-\alpha$, which is equivalent to that $\bar{y}_i > \frac{\underset{\sim}{x}'\underset{\sim}{\beta}-\theta_0}{1-\alpha}+\underset{\sim}{x}'_i\underset{\sim}{\beta}\equiv c_i$. Hence, by Lemma A.2, we have

$$P_i\left\{\hat{\alpha}(i)<1, (\hat{\alpha}(i)-\alpha)(\underset{\sim}{x}'_i\underset{\sim}{\beta}-\bar{y}_i) < \frac{\theta_0-\psi_i(\bar{y}_i)}{2}\right\}$$

$$\leq P_i\{\hat{\alpha}(i)-\alpha<c(\bar{y}_i)\}$$

$$\leq \exp\left\{-\frac{k(m-1)}{2}h_1(c(\bar{y}_i),\alpha)\right\}+\exp\left\{-\frac{k-1-p}{2}h_2(c(\bar{y}_i),\alpha)\right\} \tag{3.17}$$

Replacing (3.17) into A_{22}, by Lemma A.8(c), (d), we obtain

$$A_{22}=O(k^{-1}) \tag{3.18}$$

The preceding discussions and results are summarised as a theorem as follows.

Theorem 3.1. For the normal regression models, it is assumed that Condition C holds. Then, the empirical Bayes selection rule $\underset{\sim}{d}^*$ is asymptotically optimal and $D(\underset{\sim}{d}^*) = O(k^{-1})$ as $k \longrightarrow \infty$.

4. Appendices

Certain results which are useful to study the asymptotic optimality of the empirical Bayes selection rule $\underset{\sim}{d}^*$ are presented.

Lemma A.1. (a) For a standard normal random variable Z and $c > 0$,

$$P\{Z \geq c\} \leq \frac{1}{\sqrt{2\pi}c} \exp\left(-\frac{c^2}{2}\right)$$

(b) For a random variable $S \sim \chi^2(n)$, we have

$$P\left(\frac{S}{n} - 1 \leq c\right) \leq \exp\left(-\frac{n}{2}(c - \ln(1 + c))\right) \text{for} -1 < c < 0$$

and

$$P\left(\frac{S}{n} - 1 \geq c\right) \leq \exp\left(-\frac{n}{2}(c - \ln(1 + c))\right) \text{for } c > 0$$

Note: Part (a) is from Appendix B of Pollard [9] and part (b) is from Corollary 4.1 of Gupta, Liang and Rau [1].

Lemma A.2. For the random variable $\hat{\alpha}(i)$ defined previously, we have

(a) $P\{\hat{\alpha}(i) - \alpha > c\}$

$$\begin{cases} = 0 & \text{if } c > 1 - \alpha \\ \leq \exp\left\{-\frac{k(m-1)}{2} h_1(c, \alpha)\right\} + \exp\left\{-\frac{k-1-p}{2} h_2(c, \alpha)\right\} & \text{if } 0 < c \leq 1 - \alpha \end{cases}$$

where

$$h_1(c, \alpha) = \frac{c}{2\alpha} - \ln\left(1 + \frac{c}{2\alpha}\right)$$

and

$$h_2(c, \alpha) = -\frac{c}{2(\alpha + c)} - \ln\left(1 - \frac{c}{2(\alpha + c)}\right)$$

(b) $P\{\hat{\alpha}(i) - \alpha < c\}$

$$\begin{cases} = 0 & \text{if } c \leq -\alpha \\ \leq \exp\left\{-\frac{k(m-1)}{2} h_1(c, \alpha)\right\} + \exp\left\{-\frac{k-1-p}{2} h_2(c, \alpha)\right\} & \text{if } -\alpha < c < 0 \end{cases}$$

(c) $P\{\hat{\alpha}(i) = 1\} = P\{\hat{\alpha}(i) - \alpha = 1 - \alpha\}$

$$\leq P\{\hat{\alpha}(i) - \alpha \geq 1 - \alpha\}$$

Proof. By the definition of $\hat{\alpha}(i)$ and by an application of Lemma A.1(b), straightforward computation will lead the results.

Lemma A.3. Under Condition C(1), we have

(a) $$0 < \int_{\bar{y}_i = -\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)]\, g_i(\bar{y}_i)\, d\bar{y}_i \le M_1, \text{ and}$$

(b) $$0 < \int_{\bar{y}_i = a_i}^{\infty} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i) d\bar{y}_i \le M_1$$

for all $i = 1, \dots, k$, where M_1 is independent of k.

Proof. Straightforward computation will yield the results. Hence the details are omitted.

Lemma A.4. For $c > 0$, we have

(a) $$P_i\{\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta} > c\} \le \frac{\sqrt{b_i v^2}}{\sqrt{2\pi} c} \exp\left\{-\frac{c^2}{2b_i v^2}\right\}$$

where $b_i = \underset{\sim}{x}'_i (\underset{\sim}{x}(i) \underset{\sim}{x}'(i))^{-1} \underset{\sim}{x}_i$ and $v^2 = \frac{\sigma^2}{m} + \tau^2$

(b) $$P_i\{\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta} < -c\} \le \frac{\sqrt{b_i v^2}}{\sqrt{2\pi} c} \exp\left\{-\frac{c^2}{2b_i v^2}\right\}$$

Proof. This is a direct application of Lemma A.1(a) by noting that $\underset{\sim}{x}'_i \hat{\underset{\sim}{\beta}}(i) - \underset{\sim}{x}'_i \underset{\sim}{\beta} \sim N(0, b_i v^2)$.

Lemma A.5. Under Condition C, for sufficiently large k,

$$\underset{\sim}{x}'_i (\underset{\sim}{x}(i) \underset{\sim}{x}'(i))^{-1} \underset{\sim}{x}_i \le \frac{M_2}{k} \text{ for some } M_2 > 0 \text{ for each } i = 1, \dots, k$$

where M_2 is independent of k.

Proof. Note that $\underset{\sim}{x}\underset{\sim}{x}' = \underset{\sim}{x}(i)\underset{\sim}{x}'(i) + \underset{\sim}{x}_i \underset{\sim}{x}'_i$. Hence,

$$\frac{1}{k}\underset{\sim}{x}\underset{\sim}{x}' = \frac{1}{k}\underset{\sim}{x}(i)\underset{\sim}{x}'(i) + \frac{1}{k}\underset{\sim}{x}_i \underset{\sim}{x}'_i$$

where under Condition C(1), $\frac{1}{k}\underset{\sim}{x}_i \underset{\sim}{x}'_i \longrightarrow 0$ uniformly for each $i = 1, \dots, k$. Therefore, by Condition C(2), $\frac{1}{k}\underset{\sim}{x}(i)\underset{\sim}{x}'(i)$ converges to A for each

$i = 1, \ldots, k$. Also, since $\left(\frac{1}{k}\underset{\sim}{x}(i)\underset{\sim}{x}'(i)\right)\left(\frac{1}{k}\underset{\sim}{x}(i)\underset{\sim}{x}'(i)\right)^{-1} = I$, $k(\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}$ converges to A^{-1} for every $i = 1, \ldots, k$, and $\underset{\sim}{x}'_i k(\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}\underset{\sim}{x}_i$ converges, as $k \longrightarrow \infty$, to $\underset{\sim}{x}'_i A^{-1}\underset{\sim}{x}_i$, which are bounded uniformly for all $i = 1, \ldots, k$, under Condition C(1). That is, $0 \le \underset{\sim}{x}'_i A^{-1}\underset{\sim}{x}_i \le M_2/2$ for all $i = 1, \ldots, k$. Therefore, for sufficiently large k,

$$\begin{aligned}\underset{\sim}{x}'_i(\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}\underset{\sim}{x}_i &= \frac{1}{k}[\underset{\sim}{x}'_i k(\underset{\sim}{x}(i)\underset{\sim}{x}'(i))^{-1}\underset{\sim}{x}_i] \\ &\le \frac{2}{k}\underset{\sim}{x}'_i A^{-1}\underset{\sim}{x}_i \\ &\le \frac{M_2}{k}\end{aligned}$$

Lemma A.6. (a) For $0 \le t_1 < t_2 < \infty$ and $n > 0$

$$\int_{t_1}^{t_2} x \exp\{-n[x - \ln(1+x)]\}\,dx = O(n^{-1})$$

(b) For $0 \le t_1 < t_2 < 1$ and $n > 0$

$$\int_{t_1}^{t_2} x \exp\{n[x + \ln(1-x)]\}\,dx = O(n^{-1})$$

Proof. The result can be obtained through direct computation. The detail is omitted here.

For $-\infty \le t_1 < t_2 \le a_i$, define

$$B_1(t_1, t_2, k) = \int_{t_1}^{t_2}[\theta_0 - \psi_i(\bar{y}_i)]\,g_i(\bar{y}_i)\exp\left\{-\frac{k(m-1)}{2}h_1(c(\bar{y}_i), \alpha)\right\}d\bar{y}_i$$

$$B_2(t_1, t_2, k) = \int_{t_1}^{t_2}[\theta_0 - \psi_i(\bar{y}_i)]\,g_i(\bar{y}_i)\exp\left\{-\frac{k-1-p}{2}h_2(c(\bar{y}_i), \alpha)\right\}d\bar{y}_i$$

Then, the following results are obtained.

Lemma A.7.

$$\text{(a)}\quad B_1(-\infty, a_i, k) = \begin{cases} O\left(\exp\left\{-\frac{k(m-1)}{2}h_1\left(\frac{1-\alpha}{2}, \alpha\right)\right\}\right) & \text{if } \underset{\sim}{x}'_i\underset{\sim}{\beta} - \theta_0 = 0 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i\underset{\sim}{\beta} - \theta_0 > 0\end{cases}$$

(b) $$B_2(-\infty, a_i, k) = \begin{cases} O(\exp\{-\frac{k-1-p}{2} h_2(\frac{1-\alpha}{2}, \alpha)\}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 00 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0 \end{cases}$$

Proof. (a) As $\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 0, c(\bar{y}_i) = \frac{\theta_0 - \underset{\sim}{x}'_i \underset{\sim}{\beta}}{2(\underset{\sim}{x}' \underset{\sim}{\beta} - \bar{y}_i)} + \frac{1-\alpha}{2} = \frac{1-\alpha}{2}$ for $\bar{y}_i \le a_i$. Thus, $h_1(c(\bar{y}_i), \alpha) = h_1\left(\frac{1-\alpha}{2}, \alpha\right) = \frac{1-\alpha}{4\alpha} - \ln\left(1 + \frac{1-\alpha}{4\alpha}\right) > 0.$ Hence,

$$B_1(-\infty, a_i, k) = \exp\left\{\frac{-k(m-1)}{2} h_1\left(\frac{1-\alpha}{2}, \alpha\right)\right\} \int_{-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)] g_i(\bar{y}_i) d\bar{y}_i$$

$$= O\left(\exp\left\{-\frac{k(m-1)}{2} h_1\left(\frac{1-\alpha}{2}, \alpha\right)\right\}\right)$$

since $0 \le \int_{-\infty}^{a_i} [\theta_0 - \psi_i(\bar{y}_i)] g_i(\bar{y}_i) d\bar{y}_i \le \theta_0 < \infty$

When $\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0$, define

$$a_i^* = \begin{cases} \theta_0 & \text{if } \theta_0 < a_i \\ a_i - \left(\left[\frac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - a_i}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0}\right] + 1\right) \frac{(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0)}{(1-\alpha)} & \text{if } a_i \le \theta_0 \end{cases}$$

where $[y]$ denotes the greatest integer not larger than y. Then,

$$B_1(-\infty, a_i, k) = B_1(-\infty, a_i^*, k) + B_1(a_i^*, a_i, k) \tag{A.1}$$

Since $c(\bar{y}_i)$ is decreasing in $\bar{y}_i, c(\bar{y}_i) \ge c(a_i^*)$ for $\bar{y}_i \le a_i^*$. Thus $h_1(c(\bar{y}_i), \alpha) \ge h_1(c(a_i^*), \alpha) > 0$. Therefore,

$$B_1(-\infty, a_i^*, k)$$

$$\le \exp\left\{-\frac{k(m-1)}{2} h_1(c(a_i^*), \alpha)\right\} \int_{-\infty}^{a_i^*} [\theta_0 - \psi_i(\bar{y}_i)] g_i(\bar{y}_i) d\bar{y}_i$$

$$= O\left(\exp\left\{-\frac{k(m-1)}{2} h_1(c(a_i^*), \alpha)\right\}\right) \tag{A.2}$$

Let $z \equiv z(\bar{y}_i) = \dfrac{c(\bar{y}_i)}{2\alpha} = \dfrac{\theta_0 - \psi_i(\bar{y}_i)}{4\alpha(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)} = \dfrac{\theta_0 - \underset{\sim}{x}'_i \underset{\sim}{\beta}}{4\alpha(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)} + \dfrac{1-\alpha}{4\alpha}$. Note that z is decreasing in $\bar{y}_i$ for $\bar{y}_i < a_i$ and $dz(\bar{y}_i) = \dfrac{(\theta_0 - \underset{\sim}{x}'_i \underset{\sim}{\beta})}{4\alpha(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)^2}\, d\bar{y}_i$. Straightforward computation yields that

$$B_1(a_i^*, a_i, k)$$

$$= \int_{a_i^*}^{a_i} \frac{16\alpha^2 (\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)^3 g_i(\bar{y}_i)}{(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0)} z(\bar{y}_i) \exp\left\{-\frac{k(m-1)}{2} h_1(c(\bar{y}_i), \alpha)\right\} d(-z(\bar{y}_i)) \tag{A.3}$$

Since $\bar{Y}_i - \underset{\sim}{x}'_i \underset{\sim}{\beta} \sim N\left(0, \dfrac{\sigma^2}{m} + \tau^2\right)$, $16\alpha^2 (\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i)^2 g_i(\bar{y}_i) \le c_1$ for some positive value c_1 for any $\bar{y}_i$. When $\theta_0 < a_i < \underset{\sim}{x}'_i \underset{\sim}{\beta}$, for $\theta_0 \equiv a_i^* \le \bar{y}_i \le a_i$, $0 \le \dfrac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i}{\underset{\sim}{x}' \underset{\sim}{\beta} - \theta_0} \le 1$. When $a_i \le \theta_0 < \underset{\sim}{x}'_i \underset{\sim}{\beta}$, for $a_i^* \le \bar{y}_i \le a_i$, by the definition of a_i^* and noting that $\dfrac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - a_i}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0} = \dfrac{1}{1-\alpha}$, we have

$$\begin{aligned}
0 &\le \frac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \bar{y}_i}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0} \\
&\le \frac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - a_i^*}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0} \\
&= \frac{(\underset{\sim}{x}'_i \underset{\sim}{\beta} - a_i) + (a_i - a_i^*)}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0} \\
&= \frac{1}{1-\alpha} + \frac{\left(\left[\dfrac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - a_i}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0}\right] + 1\right) \dfrac{(\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0)}{(1-\alpha)}}{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0} \\
&= \frac{1}{1-\alpha} + \frac{1}{1-\alpha}\left(\left[\frac{1}{1-\alpha}\right] + 1\right) \equiv \alpha_1
\end{aligned} \tag{A.4}$$

where $\alpha_1 > \dfrac{1}{1-\alpha} > 1$

Plugging the preceding results into (A.3), we obtain

$$
\begin{aligned}
& B_1(a_i^*, a_i, k) \\
& \le \int_{a_i^*}^{a_i} c_1 \alpha_1 z(\bar{y}_i) \exp\left\{-\frac{k(m-1)}{2} h_1(c(\bar{y}_i), \alpha)\right\} d(-z(\bar{y}_i)) \\
& = c_1 \alpha_1 \int_{z(a_i)}^{z(a_i^*)} z \exp\left\{-\frac{k(m-1)}{2}[z - \ln(1+z)]\right\} dz \\
& = O(k^{-1}) \qquad \text{(A.5)}
\end{aligned}
$$

by Lemma A.6(a) and by noting that $0 = z(a_i) < z(a_i^*) < \infty$.

Combining (A.1), (A.2) and (A.5), it is concluded that

$$B_1(-\infty, a_i, k) = O(k^{-1})$$

Therefore, the proof of part (a) is complete.

Following an argument analogous to that of part (a) and by an application of the inequality of Lemma A.6(b), the result of part (b) can also be obtained. The detail is omitted here.

Note : By (A.4), we see that $c(a_i^*)$ and $z(a_i^*)$ are constants, independent of i. Therefore, the rate of covergence reported in Lemma A.7 holds uniformly for all $i = 1, \ldots, k$.

For $a_i \le t_1 < t_2 < \infty$, define

$$B_3(t_1, t_2, k) = \int_{t_1}^{t_2} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i) \exp\left\{-\frac{k(m-1)}{2} h_1(c(\bar{y}_i), \alpha)\right\} d\bar{y}_i$$

$$B_4(t_1, t_2, k) = \int_{t_1}^{t_2} [\psi_i(\bar{y}_i) - \theta_0]\, g_i(\bar{y}_i) \exp\left\{-\frac{k-1-p}{2} h_2(c(\bar{y}_i), \alpha)\right\} d\bar{y}_i$$

Similar to Lemma A.7, one can obtain the following results:

Lemma A.8. For $e_i \equiv \dfrac{\theta_0 + \alpha \underset{\sim}{x}'_i \underset{\sim}{\beta}}{1+\alpha}$, $c_i \equiv \dfrac{\underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0}{1-\alpha} + \underset{\sim}{x}'_i \underset{\sim}{\beta}$

(a) $$B_3(a_i, e_i, k) = \begin{cases} O\left(\exp\left\{-\dfrac{k(m-1)}{2} h_1\left(\dfrac{1-\alpha}{2}, \alpha\right)\right\}\right) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 0 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0 \end{cases}$$

(b) $$B_4(a_i, e_i, k) = \begin{cases} O\left(\exp\left\{-\frac{k-1-p}{2} h_2\left(\frac{1-\alpha}{2}, \alpha\right)\right\}\right) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 0 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0 \end{cases}$$

(c) $$B_3(c_i, \infty, k) = \begin{cases} O\left(\exp\left\{-\frac{k(m-1)}{2} h_1\left(\frac{1-\alpha}{2}, \alpha\right)\right\}\right) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 0 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0 \end{cases}$$

(d) $$\{B_4(c_i, \infty, k) = \begin{cases} O\left(\exp\left\{-\frac{k-1-p}{2} h_2\left(\frac{1-\alpha}{2}, \alpha\right)\right\}\right) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 = 0 \\ O(k^{-1}) & \text{if } \underset{\sim}{x}'_i \underset{\sim}{\beta} - \theta_0 > 0 \end{cases}$$

REFERENCES

[1] Gupta, S.S., Liang, T. and Rau, Re-Bin., 1994. Empirical Bayes rules for selecting the best normal population compared with a control. *Statistics & Decisions*, **12**, 125-147.

[2] Gupta, S.S. and Miescke, K.J., 1985. Minimax multiple t-test for comparing k normal populations with a control. *J. Statist. Plann. Inf.*, **12**, 161-169.

[3] Gupta, S.S. and Panchapakesan, S., 1979. *Multiple Decision Procedures*. John Wiley and Sons, New York.

[4] Gupta, S.S. and Panchapakesan, S., 1985. Subset selection procedures: review and assessment. *Amer. J. Math. Management Sci.*, **5**, 235-311.

[5] Gupta, S. S. and Sobel, M., 1958. On selecting a subset which contains all populations better than a standard. *Ann. Math. Statist.*, **29**, 235-244.

[6] Huang, W.T., 1975. Bayes approach to problem of partitioning k normal populations. *Bull. Inst. Math. Acad. Sinica*, **3**, 87-97.

[7] Miescke, K. J., 1981. Γ-minimax selection procedures in simultaneous testing problems. *Ann. Statist.*, **9**, 215-220.

[8] Paulson, E., 1952. On the comparison of several experimental categories with a control. *Ann. Math. Statist.*, **23**, 239-246.

[9] Pollard, David, 1984. *Convergence of Stochastic Processes*. Springer-Verlag, New York.

[10] Randles, R.H. and Hollander, M., 1971. Γ-minimax selection procedures in treatments versus control problems. *Ann. Math. Statist.*, **42**, 330-341.